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THE ANALYSIS OF REINFORCED CONCRETE FLAT SLABS BY PLASTIC THEORY

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Abstract

Within the context of plastic theory the collapse behaviour of concrete flat slabs (flat plate floor slabs) was analysed with the aid of the yield-line theory (though without taking account of shear and rotational capacity). It was found that, depending on the distribution of the reinforcement, the strength of the region around the column often constitutes the governing criterion as opposed to the collapse mechanism in the span of the slab ("span mechanism"). Better fundamental insight into the strength developed around the column ("column head mechanism") is obtained on the basis of an upper and lower bound analysis for a circular slab supported on a round column. For such a slab it proved possible to determine the exact collapse load (with its associated mechanism and distribution of bending moments) and thus to obtain a good conception of the collapse behaviour of a flat slab structure. Another aspect is the occurrence of horizontal restraint forces (dome effect) in conjunction with column head mechanisms, as a result of which the strength that can be developed around columns is favourable affected.

The analysis of reinforced concrete flat slabs by plastic theory

1 Introduction

Because of its basically very simple structural system, the flat plate floor slab of reinforced concrete, commonly referred to as a "flat slab" and consisting of a slab directly supported on columns, is a form of construction which is being more and more extensively used, particularly in industrial architecture. Of major importance in regard to such floors is the connection between the slab and the column on account of the concentration of bending moments as well as shear forces in that part of the structure.

There are various theories for determining the distribution of forces in a flat slab floor. In principle, a distinction can be drawn between elastic theory and plastic theory. In general, an analysis based on elastic theory gives a good insight into the force distribution in the slab under working load conditions, while plastic theory enables the load capacity to be determined under ultimate load conditions (failure). This report will be concerned only with application of plastic theory to the calculation of the loadbearing capacity of flat slabs, with special emphasis on the analysis of the strength of the part directly surrounding the slab-to-column connection.

In applying plastic theory a distinction is drawn between the well known upper bound and lower bound analysis respectively. For slabs the upper bound analysis is familiar as the yield-line theory, a method in which the ultimate load is determined on the basis of an assumed collapse mechanism.

An important limitation imposed in this report is that only the flexural strength of the slabs will be considered. The effect of shear force and rotational capacity will be left out of account. In this way the problem is rendered more amenable to analysis by means of plastic theory. Of course, this raises the question whether such an assumption is justified and what conditions must be satisfied in order that the analysis will give results which are representative of the actual behaviour of the structure.

In order to answer this question it will be examined how the problem of clarifying the collapse behaviour around the column support in a flat slab is generally approached. As a rule, for this purpose, experiments are performed on simply-supported circular slabs with the load applied to a column-shaped structural feature at the centre thereof.

The circular test specimen is assumed to correspond to that part of a flat slab which is bounded by the line of zero bending moment (contraflexure) in the span of the floor. The results of the experiments are considered to be representative of the strength of the actual structure around the column.

The experiments show that several collapse modes are possible, depending on the quantity of reinforcement in the slab (see also [1]).

Fig. 1 schematically indicates how the collapse load is affected by the percentage of reinforcement. Characteristic of the collapse behaviour for low reinforcement percen-

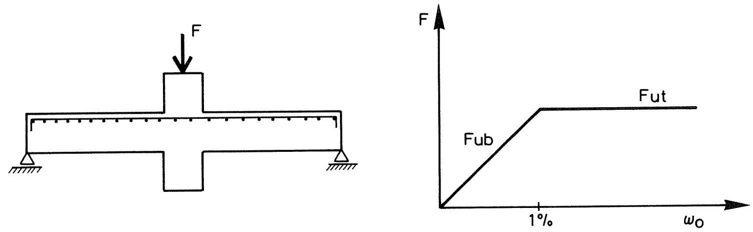


Fig. 1. Effect of reinforcement percentage on collapse load.

tages is that yielding of the reinforcement occurs throughout the slab, so that a collapse mode develops which extends to the edges of the slab. The deformation capacity of these slabs is generally considerable.

With high reinforcement percentages a collapse mode occurs in which yielding of the reinforcement is confined to a zone close to the column or in which no yielding of the reinforcement occurs at all. Collapse in accordance with a mechanism of this kind is usually referred to a case of shear failure (punching). This mechanism is characterized by its brittle behaviour. The magnitude of the load at which shear failure takes place is virtually independent of the quantity of reinforcement in the slab.

It is to be noted that the flexural strength of the slab, i.e., its resistance capacity with regard to bending, is determined on the basis of a simple collapse mechanism. With low reinforcement percentages there is found to be good agreement between the theoretically predicted collapse load and the test results obtained (see [1]). As for slabs containing higher percentages of reinforcement it must be stated that there is as yet no satisfactory method of analysis for calculating the maximum load that a circular slab can support.

Of course, over the years a number of theoretical approaches to calculating the strength of flat slabs around columns have been developed by various investigators, including Kinnunen and Nylander [2], [3], Braestrup [4], and Long and Bond [5]. In these treatments of the subject the great problem always consists in the formulation of a suitable failure criterion for the concrete in the vicinity of the column head (concrete cracked in a triaxial state of stress, with concentrated load acting in the compressive zone of the concrete). Another problem is that of how the effect of (shear) reinforcement can be introduced in a theoretically correct manner.

In the light of the above considerations it can be stated that flexural failure will in general occur if the slab contains a low percentage of reinforcement. For slabs fulfilling this condition the application of plastic theory can provide a point of departure for obtaining insight into the loadbearing capacity of these floors [6]. An interesting aspect is that, with plastic analysis and more particularly with yield-line analysis, the strength around the column will, in this present treatment of the subject, be determined on the basis of the overall loadbearing capacity of the structure, i.e., abandoning the principle of basing the analysis on a circular slab, an approach in which any form of redistribution of forces is ruled out.

Finally, it is to be noted that the problem of the strength of flat slabs around columns, for a combined state of bending and shear force, is being studied by TNO-IBBC, since it is characteristic of flat slabs that at the section around a column there occurs, besides bending, a substantial shear force which may adversely affect the strength.

In addition to the upper bound analysis a lower bound analysis will be carried out in order to obtain closer insight into the failure behaviour around the column. The analysis relates to a circular slab, clamped but unsupported at its edge and centrally supported by a circular column. It was found possible, for this slab, to determine the exact collapse load with the associated mechanism and distribution of bending moment [7].

Finally, the phenomenon of the occurrence of horizontal normal forces which affect the strength of a flat slab around a column will be investigated. More particularly, this strength is favourably influenced by these restraint forces (dome effect) [8].

The principal points of departure for the treatment of the subject are that the material is schematized as displaying rigid-plastic behaviour and that tensile stress in the concrete is neglected. The stress-strain (σ-ε) diagrams of the concrete and of the steel are shown in Fig. 2.

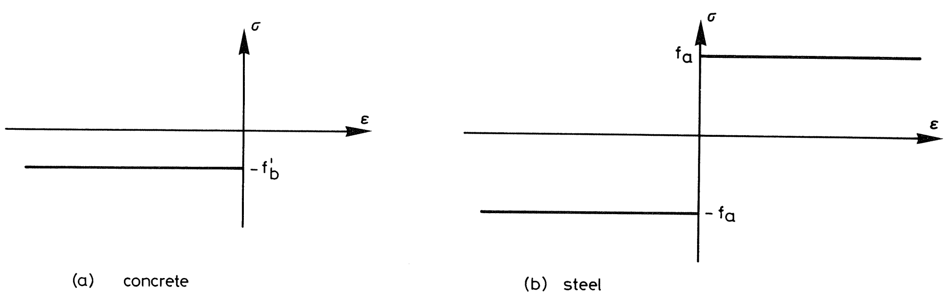


Fig. 2. σ-ε diagrams of concrete (a) and steel (b).

The material is assumed to be homogeneous, the concrete being isotropically reinforced, with the quantity of top reinforcement not equal to the quantity of bottom reinforcement. For the sake of simplicity a flat slab with square panels will be considered, supported by round (circular-section) columns and carrying uniformly distributed load on the entire area of the slab. In principle, only the case of an ideal inner panel of such a floor will be dealt with. The situation with regard to edge and corner panels will merely be considered in passing.

2 Yield-line theory for flat slabs

Within the scope of plastic theory the maximum strength (loadbearing capacity) of flat slabs can, in the first place, be determined with the aid of yield-line analysis. This gives an upper bound solution for the collapse load. The ultimate strength of the structure can be calculated on the basis of a – geometrically possible – collapse mechanism, the collapse load being obtained by equating the work done by the external load to the work done internally in the yield lines (Prager's first theorem). For determining the lowest collapse load we must seek the most unfavourable possible collapse mode, so that the load thus found is either too high or is exact.

The magnitude of the collapse load can in principle be calculated in two ways, namely, using the work method or alternatively using the equilibrium approach. In the first-mentioned method the total work done by the external load is equated to the sum of the work done (dissipated energy) in the yield lines. With the equilibrium approach the bending moment equilibrium is established for each element ("fragment") of the slab. In this way a number of equations are obtained from which the collapse load can be solved. In the present treatment of the subject only the work method will be employed.

For slabs with isotropic reinforcement the yield-line theory generally leads to simple expressions for the collapse load. In such slabs the reinforcement is so arranged that the same properties are obtained in all directions. With unequal quantities of top and bottom reinforcement, respectively, the collapse load can be expressed in the positive and the negative fully plastic moment, designated by mp and mp' respectively (these are moments per unit length). For the material behaviour assumed here, the fully plastic moment is:

$$mp = \tfrac{1}{2}\omega_a f_a h^2 \left(2 - \frac{\omega_a f_a}{f_b'}\right) \qquad (1)$$

where:

mp = fully plastic moment per unit length
ω_a = reinforcement percentage
h = effective depth
f_a = design value for tensile strength of reinforcement
f_b' = design value for compressive strength of concrete

First, the application of the yield-line theory to an ideal inner region of a flat slab with square panels, supported by round columns, will be examined. Special attention will be paid to the effect of the column dimension on the collapse mechanism and on the collapse load of the flat slab.

In principle, two possible collapse modes for a uniformly loaded flat slab must be considered, namely, a "span mechanism" and a "column head mechanism" (Fig. 3).

In the span mechanism (Fig. 3a) the yield-line pattern comprises a (straight) negative yield line along the edge of the column and a positive yield line at mid-span.

With the aid of the work method the collapse load can be determined as a function of the column dimension:

$$q = \frac{8(mp + mp')}{a^2} \frac{1}{(1-c)^2} \qquad (2)$$

where:

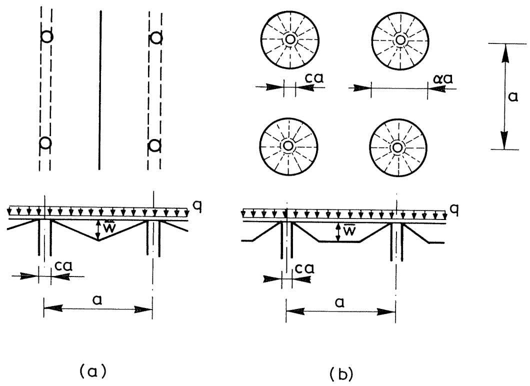


Fig. 3. Span mechanism (a) and column head mechanism (b) for an ideal inner panel of a flat slab.

q = uniformly distributed load
mp = positive fully plastic moment
mp' = negative fully plastic moment
a = centre-to-centre spacing of columns
c = ratio of column diameter to column spacing

If the reinforcement in the slab is not uniformly distributed but is instead, for example, more closely concentrated in a column strip than in a middle strip, then the fully plastic moment should be conceived as an average value along the yield line.

A graphic representation of the collapse load as a function of the column dimension is given in Fig. 4a. This diagram shows the collapse load for the range of column dimensions which is of practical importance ($c \leqslant 0.15$).

The yield-line pattern associated with the second failure mode, namely, the column head mechanism, comprises a negative (circular) yield line alone the circumference of the column, negative radial yield lines, and a positive circular yield line in the span at a certain distance from the column. The location of the last-mentioned yield line is introduced as a parameter into the calculation and is expressed in the value α, this being the ratio of the diameter of the yield-line region and the centre-to-centre spacing of the columns. The extent of the yield-line region will be so determined that the collapse load is a minimum.

The following expression for the collapse load is obtained from a calculation using the work method (see Appendix A):

$$q = \frac{2\pi\,(mp + mp')}{a^2} \left\{ \frac{\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \right\} \tag{3}$$

From $\partial q/\partial\alpha$ it follows that the optimum for the value α is:

$$\alpha = \sqrt[3]{(12/\pi - c^2)c/2} \tag{4}$$

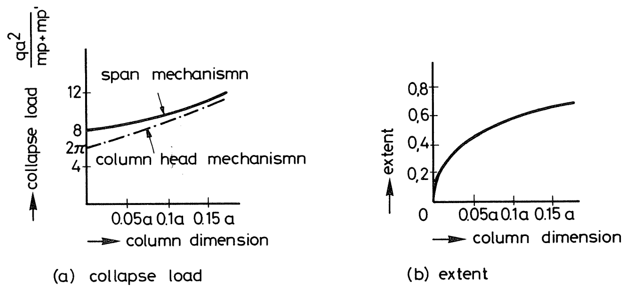


Fig. 4. Collapse load for span mechanism and column head mechanism, and extent of column head mechanism, as functions of column dimension.

The lowest value of the collapse load for the column head mechanism is indicated in Fig. 4a, while the extent of the yield-line region, expressed in α, is indicated in Fig. 4b.

It emerges that for isotropic reinforcement the extent of the yield-line region is independent of the quantity of reinforcement in the slab. The column head mechanism is found to have a finite dimension so long as the column dimension has a value greater than zero. As known, in the limiting case of a flat slab ideally carried on point-type supports ($c \rightarrow 0$) the dimension of the star-shaped pattern is zero. With columns having small dimensions a small increase in column dimension produces a considerable increase in the dimension of the column head mechanism. For practical values of the column dimension ($c = 0.06$ to 0.08) the extent of the column head mechanism corresponds approximately to the size of the column strip in the flat slab. The column strip can for this purpose be defined as a slab strip, extending over the columns and having a width which is usually taken as equal to half the span of the flat slab structure.

An investigation of the strength of the two collapse mechanisms (see Fig. 4) shows that both for the span mechanism and for the column head mechanism a substantial increase in strength occurs when the column dimension increases. In addition, the column head mechanism for the isotropically reinforced square-panel flat slab is found to have the lower collapse load. For a flat slab of this kind the strength around the column turns out to be the governing condition in relation to the span mechanism.

In constructional practice it will be endeavoured to suppress this column head mechanism. One possible way to do this consists in providing extra top reinforcement over the column in the above-mentioned column strip of the slab. An optimum distribution of the reinforcement in the slab can be determined by making an appropriate choice of the ratio of the top reinforcement in the column strip to that in the middle strip.

The situation where an actual flat slab floor is supported on rectangular or square columns will often occur in practice. Obviously, the analysis of the strength (loadbearing capacity) can in principle be carried out in the same way as for a flat slab supported on round columns. For an ideal inner panel of a flat slab supported on square columns the governing column head mechanism is found to be as shown in Fig. 5.

The collapse load for the column head mechanism is:

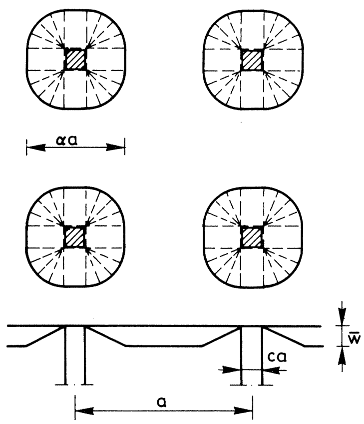


Fig. 5. Column head mechanism for an ideal inner panel of a flat slab on square columns.

$$q = \frac{2\pi\,(mp + mp')}{a^2}\left\{\frac{1 + 4c/(\alpha - c)}{1 - \alpha c - (\alpha - c)^2\pi/12}\right\} \tag{5}$$

The optimum for the value α is difficult to formulate explicitly and will have to be determined by substitution of various values into equation (5).

Besides predicting the strength for ideal inner panels, it is necessary also to bestow attention on the strength of edge and corner panels of flat slabs. For these panels a distinction will likewise be drawn between span mechanisms and column head mechanisms. By way of illustration of this field of investigation, Fig. 6 shows a span mechanism and some column head mechanisms for a slab of infinite extent in one direction, supported by square columns along its edges.

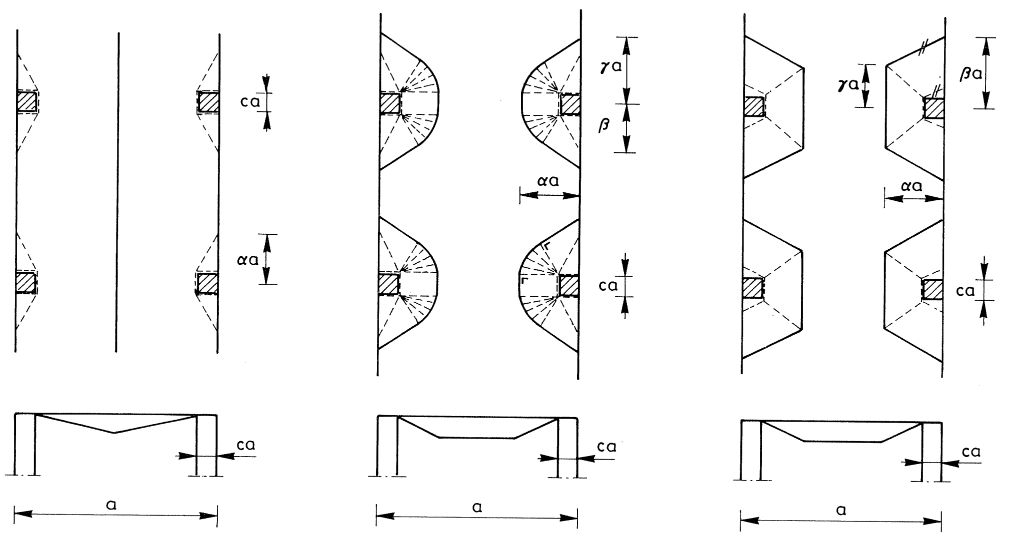


Fig. 6. Collapse mechanisms in the edge region of a flat slab: span mechanism, column head mechanism I, column head mechanism II.

In general, the expressions for the structural strength in such cases are less simple to determine than those for an ideal inner region of a flat slab. With regard to the column head mechanisms there is moreover the complication that the dimension of such a mechanism is governed by several parameters, so that optimization of the collapse load will as a rule have to be done by a numerical procedure. A highly schematized yield-line pattern will then usually be analysed (e.g., column head mechanism II instead of mechanism I as shown in Fig. 6). It is to be noted that in that case the strength will certainly be over-estimated (upper bound theorem).

To summarize, it can be said that, on the basis of the analysis of the strength of isotropically reinforced flat slabs, it emerges that when the strength is calculated with the aid of the yield-line theory it is necessary to consider a column head mechanism besides a span mechanism. For both these mechanisms the collapse load is dependent on the column dimension. In the case of the isotropically reinforced square-panel flat slab the column head mechanism gives the lower collapse load. Besides, the extent of this last-mentioned mechanism – for practical ratios of column diameter to slab span – is found to be approximately equal to the dimension of the column strip of the flat slab, so that suppression of the column head mechanism is possible by adding extra top reinforcement over the column.

So far only the analysis of simple isotropically reinforced flat slabs with square panels has been considered. It will be evident that the strength of flat slabs with non-isotropic reinforcement and/or rectangular panels can be analysed in the same way.

Finally, it is to be noted that so far only the application of the elementary yield-line theory has been considered, i.e., where only bending moments in the yield lines play a part. However, under certain conditions horizontal normal forces will also come into play in conjunction with collapse (the so-called dome effect), in consequence of which the ultimate strength is favourably affected. The dome effect associated with column head mechanism will be examined more closely in Chapter 4.

3 Lower bound analysis for flat slabs

Besides the upper bound analysis there exists, with the scope of plastic theory, an analysis procedure which gives a lower bound solution for the collapse load. For arriving at such a solution a bending moment distribution must be established which satisfies the equilibrium requirements and the boundary conditions and for which the yield criterion is nowhere exceeded in the slab. The maximum strength is calculated from the equilibrium equation applicable to the slab. The calculated maximum strength for a given slab is then either too low or is equal to the exact solution. For obtaining a lower bound solution the analysis proceeds from a safe distribution of the moments.

Obviously, if a moment distribution is adopted which deviates from the elastic moment distribution, a redistribution of moments will have to take place before the ultimate strength is reached. For the purpose of the lower bound analysis the slab is assumed to possess adequate deformation capacity.

In collapse load analysis exact solutions are known for various simple cases (for example, see [9] and [10]). For those cases it is possible to indicate a moment distribution for which the collapse load corresponds to the strength that is found on the basis of an upper bound analysis. Quite often, however, it proves difficult to determine a lower bound solution that does not deviate too much from an upper bound solution that has been found. In principle this situation occurs in the analysis of the strength of flat slabs. Even for the simple case of an ideal inner panel of such a slab with square panels it is not possible to indicate a tolerably accurate lower bound solution.

The importance of a good lower bound solution for flat slabs becomes evident on considering that - for isotropically reinforced slabs - the column head mechanism is the governing collapse mode for an ideal inner region of a flat slab. It gives rise to a "star-shaped" pattern which extends over a certain area around the column. In view of the location of the positive yield-line in the span (expressed in the value α) and in view of the elastic distribution of bending moments in the slab, with the largest positive moment at mid-span, it can be asked whether the column head mechanism is indeed the mechanism that gives the lowest collapse load.

In fact, a lower bound solution will have to be sought for the case of an ideal inner region of a flat slab supported on round columns. Since an analysis for a square ideal inner region is too complicated, however, the analysis will instead be presented for a circular slab clamped but unsupported at its edge and centrally supported by a round column (Fig. 7). Because of axial symmetry this slab is simple to analyse, while the boundary conditions correspond to those for an ideal inner panel of a flat slab. With reference to the distribution of forces in the circular slab it is thus possible to gain insight into the collapse behaviour of an ideal inner panel (see also [7]).

The upper bound solution for the collapse load of the circular slab will be given. The chosen slab dimensions and collapse mechanism are shown in Fig. 7.

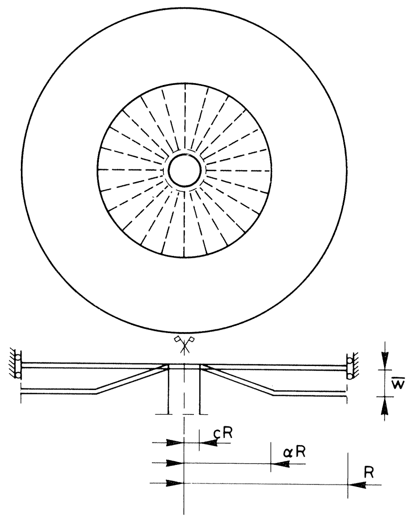


Fig. 7. Collapse mechanism of a circular slab.

With the aid of the work method (similar to the analysis presented in Appendix A) the following expression for the collapse load can be determined:

$$q = \frac{mp + mp'}{R^2} \left\{ \frac{2\alpha}{\alpha - c - (\alpha^3 - c^3)/3} \right\} \tag{6}$$

The minimum for the strength of the slab is found for a value α as expressed by:

$$\alpha = \sqrt[3]{(3 - c^2)c/2} \tag{7}$$

The extent of the yield-line region has thus in fact been determined.

Better insight into the column head mechanism can be obtained by proceeding from a lower bound analysis for the circular slab. The problems consists in establishing a moment distribution which satisfies equilibrium requirements and the yield criterion. By making use of the axial symmetry of the circular slab it is possible to obtain for this slab a lower bound solution which gives the same load as the upper bound solution. The lower bound solution is based on the assumption that the load on the slab is transferred via radially distributed shear forces to the column. By moreover making an assumption as to the configuration of either the radial or the tangential moments in various parts of the slab - twisting moments are zero because of axial symmetry - it becomes possible completely to calculate the distribution of forces with the aid of the remaining equilibrium equations.

A detailed calculation of the lower bound solution is presented in Appendix B. In Fig. 8 a lower bound solution for the tangential and radial moments is given (in the slab section on the right). Note that the moment distribution has been drawn for $c = 0{,}1$, i.e., the column diameter is equal to one-tenth of the slab diameter. For the inner region of the slab ($c \leqslant r/R \leqslant \alpha$) the following solution is obtained for the moment distribution:

$$m_{rr} = \frac{mp + mp'}{2} \left\{ \frac{r - cR}{r} - \left(\frac{r^3 - c^3R^3}{3rR^2} \right) \right\} \frac{2\alpha}{\alpha - c - (\alpha^3 - c^3)/3} - mp' \tag{8}$$

$$m_{tt} = -mp' \tag{9}$$

In the outer region of the circular slab ($\alpha \leqslant r/R \leqslant 1$) there is, in principle, an infinite number of possible solutions for the distribution of the bending moments. Two distributions have been analysed in Appendix B, which can be conceived as constituting the extreme limits for the various moment distributions. Fig. 8 shows the distribution which ties up most closely with the expected distribution at failure (see right-hand part in Fig. 8). Besides the distribution of the moments in the plastic range of behaviour, the approximate distribution in the elastic range is also indicated in Fig. 8 (left-hand part).

The distribution of the moments at failure, i.e., in the stage of collapse, can be formulated as follows for the outer region ($\alpha \leqslant r/R \leqslant 1$) of the slab:

$$m_{rr} = mp \tag{10}$$

$$m_{tt} = mp - \frac{mp + mp'}{2} \left\{ \frac{2\alpha}{\alpha - c - (\alpha^3 - c^3)/3} (1 - (r/R)^2) \right\} \tag{11}$$

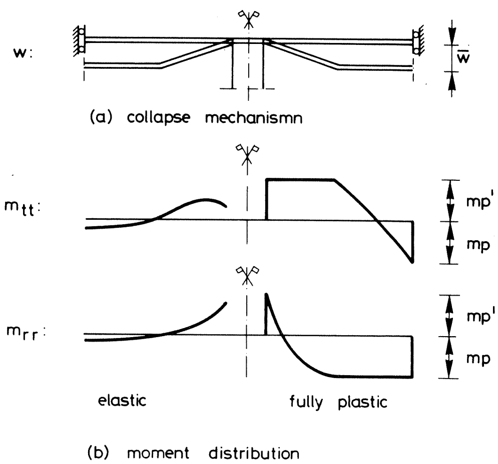


Fig. 8. Circular slab with: (a) collapse mechanism; (b) moment distribution (elastic and fully plastic).

The associated lower bound solution for uniformly distributed load on the slab is:

$$q = \frac{mp + mp'}{R^2}\left\{\frac{2\alpha}{\alpha - c - (\alpha^3 - c^3)/3}\right\} \tag{12}$$

Of course, the moment distribution is completely in agreement with the chosen failure mode in the upper bound analysis; at the negative and positive yield lines negative and positive fully plastic moments are respectively found.

In order to obtain a correct conception of the behaviour of the circular slab, the elastic distribution of the moments is shown (in the slab section of the left) in Fig. 8. Note that no significance should be attached to the absolute magnitude of the moments; only their behaviour, i.e., their distribution as represented by the curves, is of importance. As was to be expected, there occurs a considerable redistribution of moments between the elastic and the plastic range. This redistribution can explain why a local column head mechanism can develop in the slab at failure.

Since the situation in a circular slab corresponds approximately to that in an ideal inner panel of a flat slab, it can be inferred that the column head mechanism is indeed a real mechanism. For isotropically reinforced flat slabs with square panels the strength around the columns is the governing criterion in relation to the span mechanism.

No experimental verification of the development of column head mechanisms is provided by the information published in the literature. So far as is known, various experiments have been performed with a view to ascertaining the collapse behaviour of flat slabs (see, inter alia, [11] and [12]). On analysing those slabs it emerges that, as a rule, the distribution of the reinforcement employed in them was so contrived that the span mechanism was the governing feature. Another phenomenon which plays a part in connection with experiments on flat slabs and which brings about an increase in strength around the column is the action of horizontal restraint forces (dome effect). This effect could explain in part the non-occurrence of the column head mechanism in various experiments on flat slabs (see Chapter 4).

The redistribution of moments has important consequences for the distribution of forces in the flat slab at failure, more particularly as regards the action of forces around a column. It emerges that, under failure conditions, the point of zero bending moment of the radial moments is located much closer to the column than in the elastic range of behaviour. While the load is being increased, the point of zero moment moves nearer and nearer the column. This means, for example, that the bottom reinforcement of the slab must be present within a large region in the span in order to be effective for the column head mechanism. Only in a limited region around the column may the bottom reinforcement conceivably be omitted. In the theoretical case of an ideal point-type support, however, the bottom reinforcement will even have to be provided throughout the whole slab.

In order to obtain information on the strength of a flat slab, tests are often performed on circular slabs which are simply supported at the edge and are subjected to load acting on a column-shaped structural feature at the centre of the slab [1]. This circular slab is conceived as corresponding to that part of a flat slab which is bounded by the zero bending moment line located at a certain distance from the column and surrounding it. This zero moment line in the span is calculated on the basis of elastic distribution of moments in the slab. In view of boundary conditions for the circular slab its collapse behaviour is considered to be representative of the behaviour of a flat slab around a column.

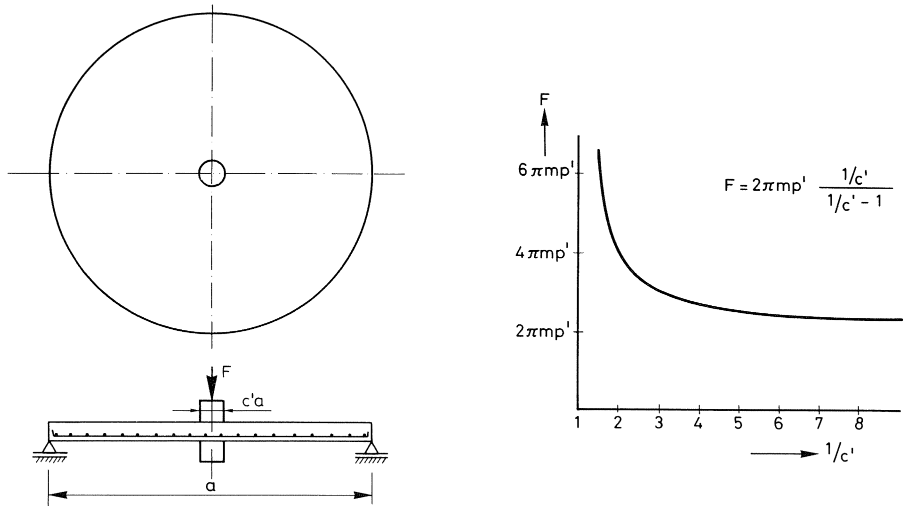
It has been established that moment redistribution produces at failure a distribution of bending moment which is manifestly different from that in the elastic range of behaviour. In consequence of this redistribution the zero moment point at failure is closer to the column. This means that the results of experiments performed on circular specimens with central supports ("waggon wheels") are not directly valid for assessing the strength of the region surrounding a column of a flat slab.

In order to obtain a clearer picture of this set of problems it will be investigated to what extent a shift in the position of the zero moment point affects the collapse load of the circular slab. For this purpose, in Fig. 9, the collapse load has been plotted as a function of the ratio of slab diameter to column diameter of the specimen.

If the collapse load of the circular slab is determined in relation to the slab/column diameter ratio ($1/c'$), the following relationship is obtained:

$$F = 2\pi m p' \frac{1/c'}{1/c' - 1} \qquad (13)$$

It is apparent from the diagram that the dimensions of the test specimen play a non-negligible part in determining the magnitude of the collapse load. It is concluded that the results of experiments on "waggon wheels" are liable to result in under-estimating the collapse load of the actual flat slab. The amount by which it is under-estimated is directly dependent on the ratio between the top and bottom reinforcement quantities in the slab. As appears from Fig. 9, it is this ratio which governs the amount of shift that the



(a) circular slab (b) collapse load F

Fig. 9. Effect of specimen dimension on collapse load F; (a) circular slab; (b) collapse load F.

zero moment point undergoes towards the column in the transitional phase between elastic behaviour and fully plastic behaviour.

To summarize, it can be stated that the experimental results obtained on circular slabs have to be correctly interpreted in order to make a valid pronouncement on the strength of the actual flat slab. The reason for the discrepancy must be sought in the fact that with "waggon wheels", because of the manner of construction, any kind of redistribution is ruled out. Besides, on account of this, the incorrect impression is created that the strength of a flat slab around a column is not affected by the bottom reinforcement in the slab. From the foregoing investigation of the subject it can be inferred that the bottom reinforcement, provided that it is correctly installed, does indeed affect the collapse load.

Finally, it is to be noted that the approach outlined above is based on idealized material properties and that only the flexural strength, i.e., the capacity to resist bending, of the slabs has been considered. It is furthermore assumed that the slab is indeed able to achieve the redistribution of bending moments envisaged here. To what extent the chosen points of departure in this approach are representative of the actual behaviour of reinforced concrete and to what extent the results are affected by neglecting the shear force are matters which will have to be further investigated.

4 Dome effect associated with column head mechanisms

It is a well-known phenomenon that reinforced concrete slabs which are so supported that the displacements at the bearings are prevented from occurring develop a degree of strength (loadbeaging capacity) which is substantially higher than that of slabs supported on bearings allowing unrestrained deformation (see also, inter alia, [13] and [14]).

This increased strength is due to the fact that, when reinforced concrete containing only a small quantity of reinforcement is subjected to pure bending, the neutral axis is situated close to the surface. Because of this, extension of the mid-depth fibre of the slab occurs; with displacement restraint at the bearings this extension is prevented from occurring. There then develops a mechanism in which the neutral axis is situated close to the mid-depth plane in the slab, with the result that horizontal compressive forces are exerted on the bearings. Since the location of the fully plastic zone now corresponds roughly to half the depth of the slab, the fully plastic moment and therefore the collapse load are relatively large.

The above-mentioned situation occurs, for example, in a slab spanning in one direction, fixed at the ends and restrained from horizontal displacement there (see Fig. 10).

Some characteristic relationships are shown in the load-displacement diagram in Fig. 10. In an elementary geometrically linear analysis the elementary collapse load (a) is obtained. If horizontal normal forces are taken into account, a higher value for the calculated loadbearing capacity is arrived at (b). As a result of the restraint forces the collapse load may, depending on the quantity of reinforcement in the slab, be substantially higher than the elementary collapse load. It is to be noted that the two relationships are both based on a rigid-plastic schematization of the behaviour of the material. If the geometric non-linear effects are taken into account, a curve such as (c) is obtained. With an elasto-plastic analysis performed with, for example, a numerical computational program a curve such as (d) will possibly be found. A characteristic feature is that, for small displacements, the load increases to above the elementary collapse load because horizontal normal forces build up in the slab. At the stage where the maximum loadbearing capacity is attained, the neutral axis is located near the centroid of the slab. With larger displacements, instability occurs and the load that can be resisted decreases because the compressive forces decrease. After the load has reached a minimum, normal forces in the reinforcement begin to play a part, so that the collapse

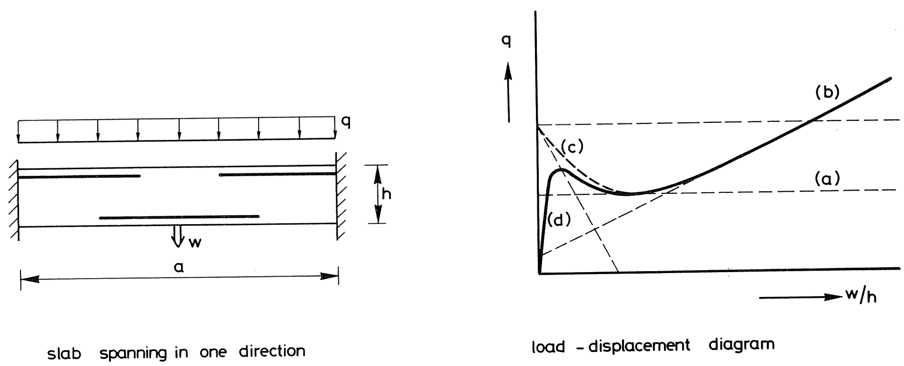


Fig. 10. Slab spanning in one direction, with load-displacement diagram: elementary collapse load (a); collapse load taking account of dome effect (b); geometrically non-linear rigid-plastic (c); elasto-plastic (d).

load increases again. Note that the peak value of the collapse load occurs at relatively low values of the deflection. The favourable effect of the normal compressive forces in the slab is usually referred as the "dome effect".

A step in the right direction towards predicting the behaviour of the load-displacement diagram, and thus the strength (loadbearing capacity) of the structure, is the calculation of the geometrically non-linear rigid-plastic branch of the diagram. In the case of a "one-way" slab, i.e., spanning in one direction only, this relationship is known (see [15]). A limitation upon the analysis based on a rigid-plastic schematization is that it provides no information on the peak of the curve. In simple case it is possible to take account of this elasto-plastic behaviour by a more theoretical approach (see, inter alia, [13] and [15]). The elastic behaviour can, for example, be incorporated in a spring system at the edge of the slab, as adopted in [16] for the slab shown in Fig. 10.

In calculating the strength of flat slabs with the aid of the yield-line theory it has emerged that, besides the span mechanism, a column head mechanism has to be distinguished. From a closer study of the column head mechanism (see Fig. 3) it appeared justifiable to assume that horizontal restraint forces are involved in this mechanism as well. This is so because, with this mechanism, the surrounding rigid region of the slab functions as a completely fixed boundary with complete restraint of horizontal displacement. Under these boundary conditions the dome effect can optimally develop.

The action of horizontal restraint forces associated with the column head mechanism in a flat slab will now be considered more closely (see also [8]). The analysis is based on a rigid-plastic schematization of the material behaviour. The slab is assumed to have equal top and bottom reinforcement, while the reinforcement is moreover assumed to be located at the extreme top and bottom fibres of the concrete section respectively (no concrete cover to the bars). In the analysis the effect of the reinforcement in the compressive zone of the concrete upon the distribution forces is neglected. Moreover, only the column head mechanisms associated with an ideal inner panel of a flat slab supported on round (circular-section) columns will be considered.

For analysing the dome effect in connection with the column head mechanism a certain portion (sector) of the circular collapse mechanism will be considered (see Fig. 11).

Suppose that, at the instant when the mechanism first develops, this sector rotates about the line AA′ and that the depth of the fully plastic zone at the circular yield line through these points is ηh. As a result of the rotation all the points which initially are located above the plane OAA′ will undergo a displacement away from the column. At the circular yield line through the points BB′ a compressive zone, of depth $(1 - \eta)h$, will be formed in the upper part of the slab in consequence of the horizontal displacement restraint. It can similarly be reasoned that at the radial yield lines a compressive zone of depth ηh will be formed on the underside of the slab. Finally, the resulting distribution of the compressive zone of the concrete at the boundary yield lines of the sector will be as shown in Fig. 11. It is clearly apparent from this distribution that, as compared with the pure bending case, additional normal forces now play a part. The above-mentioned distribution of the fully plastic zones will be adopted as the basis for a quantitative determination of the dome effect associated with the column head mechanism. The

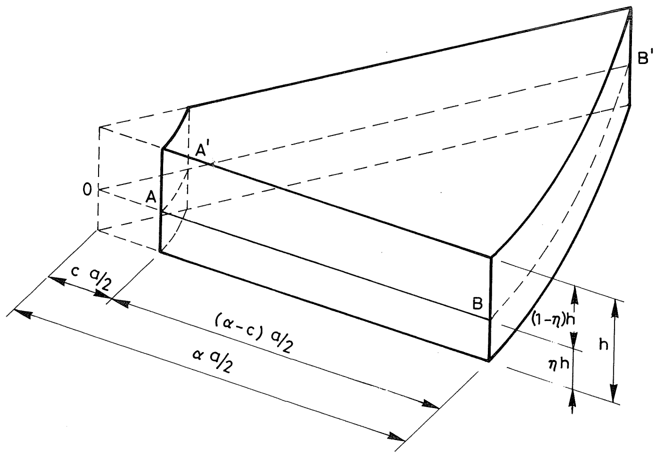


Fig. 11. Portion of the column head mechanism.

collapse load taking account of the dome effect (geometrically linear analysis) can, to start with, be calculated with the aid of the work method.

The analysis using the work method is presented in Appendix C. The following expression for the collapse load is obtained as the result thereof:

$$q = \tfrac{1}{2}(h/a)^2 f_b' \{\eta^*(2-\eta^*) + (1/2-\eta^*)^2\} \frac{4\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \tag{14}$$

where:

$\eta^* = \omega_a f_a / f_b'$ = ratio of depth of fully plastic zone to depth of slab
h = depth of slab
f_b' = design value of compressive strength of concrete
f_a = design value of tensile strength of reinforcement
ω_a = reinforcement percentage

An optimization analysis of the collapse load shows that, for the mechanism considered, the depth of the fully plastic zone at each yield line is equal to half the slab depth ($\eta = 1/2$). It can moreover be shown that the optimum value of α (determining the extent of the yield-line region) is:

$$\alpha = \sqrt[3]{(12/\pi - c^2)c/2} \tag{15}$$

This optimum is equal to the value obtained from an elementary collapse load analysis.

In the expression for the collapse load (14) the first term corresponds to the elementary portion due to the reinforcement and the associated fully plastic zone, while the effect of the horizontal restraint forces (dome effect) is incorporated in the second term. It also emerges that there is considerable similarity between the expressions for the elementary collapse load and for the collapse load with dome effect. The dome effect can be taken into account quite simply by adjustment of the value of the fully plastic moment in the collapse load formula.

For an isotropically reinforced flat slab with equal top and bottom reinforcement the following general expression is valid for the collapse load:

$$q = \frac{mp}{a^2}\left\{\frac{4\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12}\right\} \tag{16}$$

In the calculation of the elementary collapse load the following must be substituted into this expression:

$$mp = \tfrac{1}{2}(h/a)^2 f_b' \eta^*(2 - \eta^*) \tag{17}$$

while for the collapse load including dome effect the fully plastic moment is:

$$mp = \tfrac{1}{2}(h/a)^2 f_b' \{\eta^*(2 - \eta^*) - (1/2 - \eta^*)^2\} \tag{18}$$

It is to be noted that this collapse load can be conceived as an upper bound solution. A kinematically possible mechanism is adopted as the basis, and the work done by the external load is equated to the energy dissipated in the yield lines.

The next step in the analysis would consist in determining the geometrically non-linear (rigid-plastic) behaviour of the column head mechanism. This complex calculation will not be dealt with in the present article, however. Instead, merely some characteristic points of the load-displacement diagram for the column head mechanism will be indicated.

First of all, it is possible to use a virtual work approach for obtaining some idea of the slope of the load-displacement curve at the point $w/h = 0$. If the analysis is based on constant depths of the fully plastic zones and the second-order terms are taken into account, the expression thus found will correspond to the tangent to the geometrically non-linear curve at this point $w/h = 0$. This analysis is performed in Appendix C, and the result is indicated in the load-displacement diagram for the column head mechanism in Fig. 12 (for numerical data see Appendix C).

Another relation which is simple to determine is that which is obtained by considering the equilibrium of the sector, assuming that only tensile forces occur in the steel at the boundary yield lines. This situation arises if there are relatively large displacements ($w/h > 2$ to 3), when the compressive zone at the yield lines will have totally disappeared. For a specific case of a flat slab ($c = 0{,}1$ and $\omega_a = 0{,}005$) the result of such an equilibrium analysis is likewise indicated in Fig. 12.

A sufficient number of points in the load-displacement diagram are now known to obtain an idea of the (rigid-plastic) curve for the column head mechanism. It must be pointed out that the curve will have to be interpreted with due caution, however. On the basis of the calculations that have been carried out it is, for instance, not possible to indicate what magnitude the minimum for the collapse load will have and at what displacement it will occur. Nevertheless, a good idea of the geometrically non-linear effects associated with the column head mechanism is obtainable in the manner outlined above.

Finally, it is to be noted that for ascertaining the actual collapse behaviour of a flat slab around a column it is necessary to consider that the material does not in fact behave

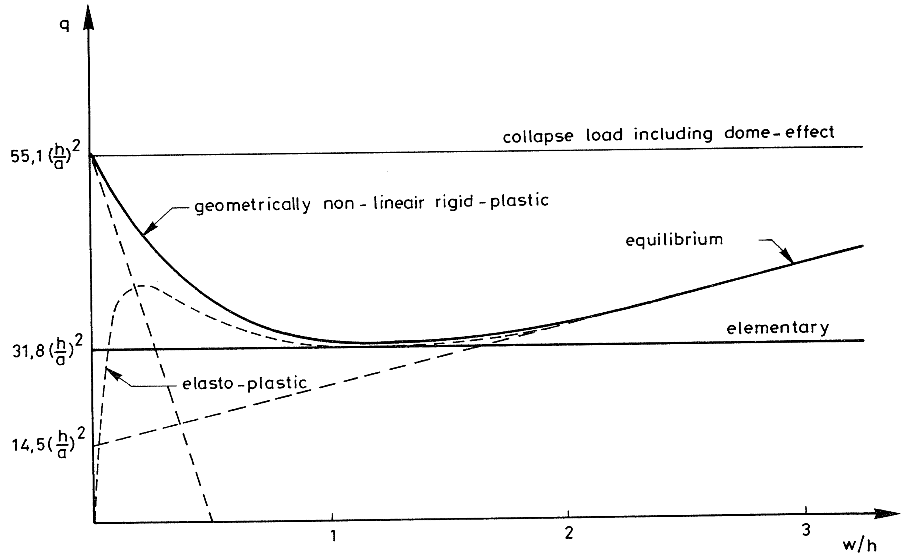


Fig. 12. Load-displacement diagram for column head mechanism ($c = 0{,}1$, $\omega_a = 0{,}005$).

in a rigid-plastic, but in an elasto-plastic manner. A possible load-displacement curve for an elasto-plastic schematization of the material behaviour has been included in Fig. 12. What has definitely been established is that, because of the action of the horizontal normal forces, the column head mechanism is found to develop substantially higher strength than that which would correspond to the collapse load calculated on the basis of an elementary analysis.

It is to be noted, however, that here the dome effect has been considered only for isotropically reinforced slabs with equal top and bottom reinforcement. Besides, the treatment of the problem has been confined to the situation in an ideal inner panel of a flat slab. The analysis can nevertheless quite practicably be extended to the inner columns of a more realistically reinforced flat slab floor structure. As a first step in that direction, the equation for the strength of an ideal inner panel of a slab with unequal top and bottom reinforcement has been derived in Appendix C. With regard to the situation at the edge and corner columns it can be stated that, due to the influence of the free edge(s), the dome effect will be relatively less than at an inner column. This being so, a closer investigation of the dome effect associated with edge and corner columns is not of much importance from the practical point of view.

5 Summary

This report deals with the application of plastic theory to determining the strength (loadbearing capacity) of flat slab structures. More particularly it is concerned with the analysis of the strength that the region around a supporting column of a flat slab can develop. A major limitation applied to the treatment of the problem in this report is that only the flexural strength of the slab is considered, while the effect of shear and rotational capacity is not taken into account.

First, the collapse behaviour of flat slabs is examined with the aid of the yield-line theory (upper bound analysis). Attention is more particularly focused on the case of an ideal inner region of a flat slab with square panels supported on round (circular-section) columns. It emerges that, besides a "span mechanism", a "column head mechanism" (collapse mode with star-shaped yield-line pattern around the column) is to be distinguished and that for an isotropically reinforced slab the strength developed around the column is the governing criterion. The maximum capacity of the two mechanisms is dependent to a great extent on the ratio of column diameter to column spacing (centre-to-centre distance). The same applies to the extent of the yield-line region associated with the column head mechanism. For practical values of the column dimension it appears that the extent of the yield-line pattern is approximately equal to the dimension of the column strip in the flat slab, so that suppression of the column mechanism by the addition of extra top reinforcement over the column is possible.

Besides the upper bound analysis there is, within the scope of plastic theory, an analysis which leads to a lower bound solution for the collapse load. For this purpose a moment distribution must be determined for which both the equilibrium requirement and the yield criterion are fulfilled. Determining a lower bound solution for an ideal inner panel of a flat slab is necessary in order to obtain a clearer fundamental insight into the column head mechanism. For this purpose an analysis is established for a circular slab clamped but unsupported at its edge and centrally supported by a round column. Because of its axial symmetry this slab is simple to analyse, while the boundary conditions correspond approximately to those for an ideal inner panel of a flat slab. It proves possible to determine the exact collapse load for this circular slab (as well as its associated mechanism and moment distribution) and thus to obtain a good conception of the collapse behaviour of a flat slab. It is found that the column head mechanism is a real mechanism and that considerable moment redistribution occurs between the stage of elastic behaviour and the fully plastic stage during load increase.

As a result of moment distribution the point of zero bending moment (contraflexure) of the radial moments in the slab is much closer to the column at failure than in the stage of elastic behaviour. While the load is being increased, the zero moment point moves closer and closer to the column. This result affects the arrangement to be adopted for the reinforcement around a column head in a flat slab. Also, the position of the moments is an important factor in structural design and in the interpretation of the results of experiments on freely supported circular slabs with a view to obtaining information on the loadbearing behaviour around a column head. These circular test specimens are conceived as corresponding to that part of a flat slab which is bounded by the zero bending moment line, around the column, in the span. Since the analysis is based on the elastic distribution of the moments, this means that, due to moment redistribution, the results of such experiments do not directly give insight into the strength of the region of a flat slab around a column. In interpreting such results it will be necessary to take account of the effect of the change in position of this zero moment point.

Finally, attention is focused on the phenomenon that in experiments on reinforced concrete slabs in which displacements within the plane of the slab are prevented at the bearings the strength (loadbearing capacity) of such slabs is found to be substantially superior to that of slabs without deformation restraint at the bearings. In that case additional normal forces are developed in the plane of the slab, so that the collapse load becomes relatively large. This phenomenon is called the dome effect.

This report deals with the dome effect associated with column head mechanism in a flat slab (ideal inner panel). The treatment of the problem is based on a rigid-plastic schematization of the material behaviour. A qualitative approximation is presented. This is followed up with an analysis of the geometrically linear collapse load including the dome effect. It emerges that the difference between the result thus obtained and the elementary collapse load can be taken into account by adjustment of the magnitude of the fully plastic moment in the collapse load formula. Next, some characteristic points of the load-displacement diagram for the column head mechanism are derived. The most probable curve representing the load-displacement behaviour (geometrically non-linear rigid-plastic) is drawn in that diagram. Finally, it is noted that for analysing the actual collapse behaviour of a flat slab it is necessary to take account of elasto-plastic instead of rigid-plastic properties. Nevertheless, because of the dome effect, a substantially higher value for the collapse load is obtained than that based on an elementary analysis.

6 Acknowledgements

This investigation of the application of plastic theory to flat slabs was undertaken in the context of final graduation studies, under the guidance of Prof. Ir. J. Witteveen and Prof. Dr. Ir. A. S. G. Bruggeling, in the Department of Civil Engineering of the Delft University of Technology. The work involved was carried out at the Institute TNO for Building Materials and Building Structures (IBBC-TNO) with the benefit of advice given by Ir. A. C. W. M. Vrouwenvelder and various other members of the staff of that organization. The author wishes to take this opportunity to express to all those concerned his thanks for the excellent guidance and advice he received in the course of his studies.

7 Notation

A	area of section
A_n	work done (normal force)
A_q	work done (uniformly distributed load)
E_d	dissipated energy
F	collapse load
F_{ub}	collapse load (yielding reinforcement)
F_{ut}	collapse load (punching shear)

M_p	fully plastic moment
N	normal force
R	radius of circular slab
a	centre-to-centre spacing of columns
c	ratio of column diameter to column spacing
c'	ratio of column diameter to slab diameter
f_a	yield stress of steel
f_b'	compressive strength of concrete
h	effective depth
m_p, m_p'	fully plastic moment (per unit length)
m_{rr}	radial moment (per unit length)
m_{rt}	twisting moment (per unit length)
m_{tt}	tangential moment (per unit length)
n_{rr}	radial normal force (per unit length)
n_{tt}	tangential normal force (per unit length)
q	uniformly distributed load
q_r	radial shear force (per unit length)
q_t	tangential shear force (per unit length)
r	radius
u	displacement
u_{tt}	tangential displacement
w	displacement
$\bar{w}$	displacement of certain point
α	ratio of column head mechanism diameter to column spacing
β, γ	parameter for extent of column head mechanism
ε	strain
η, η^*	ratio of depth of fully plastic zone to depth of slab
$\varkappa_{rr}$	radial curvation
$\varkappa_{tt}$	tangential curvation
σ	stress
θ, ϕ	angular of rotation
ω_a	reinforcement percentage

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APPENDIX A

Collapse load of column head mechanism of a flat slab

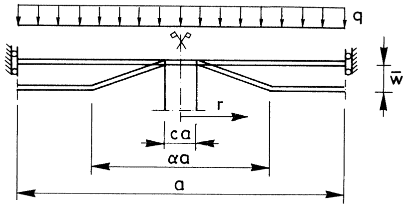

Fig. 13. Deflection diagram for column head mechanism of a flat slab.

Note: For treating the problem by the work method a particular pattern of deflection over the slab is presupposed, namely:

$$w=\bar{w}\left\{\frac{2r-ca}{\alpha a-ca}\right\} \tag{A-1}$$

where:

$$\tfrac{1}{2}ca\leqslant r\leqslant\tfrac{1}{2}\alpha a$$

The work done within the yield-line region is:

$$A_q=\int_0^{2\pi}\int_{ca/2}^{\alpha a/2} qwr\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=q\bar{w}a^2\left\{\frac{2\alpha^3-3\alpha^2c+c^3}{\alpha-c}\right\}\pi/12 \tag{A-2}$$

The work done in the outer region of the slab (uniform deflection $\bar{w}$) is:

$$A_q=q\bar{w}a^2(1-\pi\alpha^2/4) \tag{A-3}$$

The total work done is:

$$A_{\text{tot}}=q\bar{w}a^2\{(\alpha-c)-(\alpha^3-c^3)\pi/12\}/(\alpha-c) \tag{A-4}$$

For determining the dissipated energy we shall consider the energy which is dissipated in the radial yield lines and in the tangential yield lines respectively.

In the radial yield lines:

$$E_d=\int_0^{2\pi}\int_{ca/2}^{\alpha a/2} mp\varkappa_{tt}\,r\,\mathrm{d}r\,\mathrm{d}\theta \tag{A-5}$$

where:

$$\varkappa_{tt}=-1/r\frac{\partial w}{\partial r}=\frac{2\bar{w}}{(\alpha-c)ar} \tag{A-6}$$

$$E_d = 2\pi \int_{ca/2}^{\alpha a/2} mp \frac{2\bar{w}}{(\alpha - c)a} \, dr = 2\pi mp \tag{A-7}$$

In the tangential yield lines:

$$E_d = \int_0^{2\pi} mp \frac{\partial w}{\partial r} \alpha \frac{a}{2} \, d\theta + \int_0^{2\pi} mp \frac{\partial w}{\partial r} c \frac{a}{2} \, d\theta = 2\pi mp\bar{w} \frac{\alpha}{\alpha - c} + 2\pi mp\bar{w} \frac{c}{\alpha - c} \tag{A-8}$$

Total dissipated energy:

$$E_d = 2\pi mp\bar{w} \left\{ \frac{2\alpha}{\alpha - c} \right\} \tag{A-9}$$

By equating the work done to the dissipated energy the following collapse load is obtained:

$$q = \frac{mp}{a^2} \left\{ \frac{4\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \right\} \tag{A-10}$$

A minimum for the collapse load is obtained from the equation:

$$dq/d\alpha = 0$$

whence it follows that:

$$\alpha = \sqrt[3]{(12/\pi - c^2)c/2} \tag{A-11}$$

APPENDIX B

Upper and lower bound solution for circular slab

In analogy with the upper bound analysis for the collapse load associated with the circular column head mechanism in a flat slab (see Appendix A) the following expression can be derived for the collapse load associated with the column head mechanism for the circular slab:

$$q = \frac{mp + mp'}{R^2} \left\{ \frac{2\alpha}{\alpha - c - (\alpha^3 - c^3)/3} \right\} \tag{B-1}$$

and

$$\alpha = \sqrt[3]{(3 - c^2)c/2} \tag{B-2}$$

An approximate overall conception of the results is obtained by comparing them with those for an ideal inner panel (put $R = \frac{1}{2}\sqrt{\pi/4}$ a), as has been done in Table 1.

Table 1. Extent of column head mechanism and collapse load for an ideal inner panel and a circular slab.

	ideal inner pannel		circular slab	
column diameter	α	q	α	q
0	0,0	$12{,}56\,\frac{mp+mp'}{2a^2}$	0,0	$12{,}56\,\frac{mp+mp'}{2a^2}$
0,01	0,267	13,31 ”	0,247	13,37 ”
0,05	0,457	15,03 ”	0,421	15,26 ”
0,10	0,575	16,98 ”	0,531	17,46 ”
0,15	0,658	19,04 ”	0,607	19,89 ”
0,20	0,723	21,32 ”	0,666	22,62 ”

For practical column dimensions the differences are found to be relatively small, particularly with regard to collapse load of the slabs.

For the case where axial symmetry exists, and adopting polar co-ordinates in the calculation, the lower bound solution for the circular slab will be determined.

A section through the slab is shown in Fig. 14.

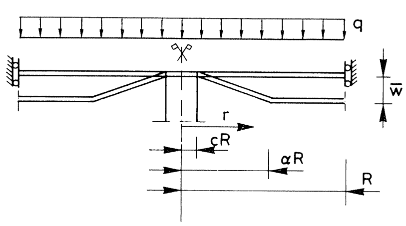

Fig. 14. Circular slab supported on a round column.

First, proceeding from the vertical equilibrium of a circular portion of the slab, the following equation for the distributed shear force in the slab can be derived:

$$q_r = q(R^2 - r^2)/2 \tag{B-3}$$

$$q_t = 0 \tag{B-4}$$

Besides, on account of symmetry, no twisting moments occur in the slab ($m_{rt} = 0$).

For an analysis of the distribution of forces it is now sufficient to solve the following equilibrium equation (having due regard to the yield conditions):

$$rq_r = \frac{\mathrm{d}(rm_{rr})}{\mathrm{d}r} - m_{tt} \tag{B-5}$$

By assuming either the radial or the tangential moment distribution it is possible completely to determine the distribution of forces in the slab.

Because of the radial yield lines expected to develop around the column, the tangential moment is first taken as equal to the negative fully plastic moment mp'.

Choose:

$$m_{tt} = -mp'$$

Integration of the equilibrium equation (B-5) gives:

$$m_{rr} = -mp + q\left(R^2 - \frac{r^2}{3}\right)/2 + C_1/r \tag{B-6}$$

The constant of integration follows from the condition that for $r = cR$ the moment $m_{rr} = -mp'$ (negative yield line around the column), so that we finally obtain for the radial moment:

$$m_{rr} = -mp' + q\left\{\frac{rR^2 - cR^3}{r} - \left(\frac{r^3 - c^3R^3}{3r}\right)\right\}/2 \tag{B-7}$$

The collapse load is determined from the condition that $m_{rr} = mp$ at a particular location in the span, denoted by the value α, whence is obtained:

$$q = \frac{mp + mp'}{R^2}\left\{\frac{2\alpha}{\alpha - c - (\alpha^3 - c^3)/3}\right\} \tag{B-8}$$

The lower bound solution for the collapse load is exactly equal to the upper bound solution for this case.

The distribution of the bending moment is then:

$$m_{rr} = \frac{mp + mp'}{R^2}\left\{\frac{r - cR}{r} - \left(\frac{r^3 - c^3R^3}{3rR^2}\right)\right\}\frac{2\alpha}{\alpha - c - (\alpha^3 - c^3)/3} - mp' \tag{B-9}$$

$$m_{tt} = -mp'\,(m_{rt} = 0)$$

The location in the span where the radial moments attain a minimum (put: $r = \alpha R$) is determined from the condition that:

$$\left(\frac{\mathrm{d}m_{rr}}{\mathrm{d}r}\right)_{r=\alpha R}=0 \tag{B-10}$$

The expression for α is obtained on working out (B-10):

$$\alpha=\sqrt[3]{(3-c^2)c/2} \tag{B-11}$$

The value α in fact indicates the location at which the positive yield line occurs in the span of the slab.

In Fig. 15 the moment distribution – according to equation (B-9) – is indicated at (a). It is to be noted that the diagram for the moment relates to the case where $c=0{,}1$ (i.e., the column diameter is one-tenth of the centre-to-centre spacing of the columns).

For the region $c \leqslant r/R \leqslant \alpha$ the moment distribution indicated here is the only correct distribution associated with the exact solution for the collapse load. In the region $\alpha \leqslant r/R \leqslant 1$ it is possible in principle to give an infinite number of solutions for the distribution of forces.

It is an obvious choice to take the radial moments in the outer region of the slab as equal to the positive fully plastic moment ($m_{rr}=mp$) and to calculate the tangential moments from the equilibrium equation (B-5). In this way the following distribution for the moments is finally obtained:

$$m_{rr}=mp$$

$$m_{tt}=mp-\frac{mp+mp'}{2}\left\{\frac{2\alpha}{\alpha-c-(\alpha^3-c^3)/3}(1-(r/R))^2\right\} \tag{B-12}$$

An approximate curve representing this distribution is included as (b) in Fig. 15.

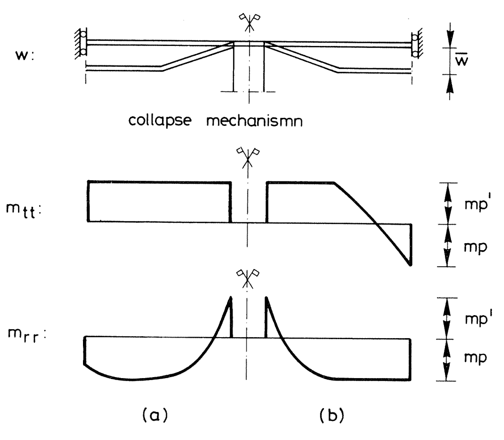

Fig. 15. Upper and lower bound solution for circular slab including collapse mechanism with two possible distributions for the moments (m_{tt} and m_{rr}).

APPENDIX C

Collapse load of column head mechanism including dome effect

First, the collapse load of the column head mechanism including the dome effect will be analysed with the aid of the work method. The analysis proceeds in the same way as that for the elementary collapse load. The term for the work done by the external load remains unchanged, while the energy dissipated in the yields lines is determined on the basis of the stress distribution shown in Fig. 11, which indicates the distribution of the stresses at the yield lines forming the boundaries of a particular portion (sector) of the column head mechanism.

Note that, for calculating the energy dissipated in the yield lines, an adjustment of the value of the fully plastic moment *mp* must be substituted into the relevant formulae. For the fully plastic moment of a yield line with a depth ηh we obtain, for example:

$$mp = \tfrac{1}{2}h^2 f_b' \eta^*(2-\eta^*) + h^2 f_b'(\eta - \eta^*)(1/2 - (\eta + \eta^*)/2) \quad \text{(C-1)}$$

On substitution of the relations concerned – equations (A-7) and (A-8) – the following expression is obtained for the total energy dissipated in the yield lines:

$$E_d = 4\pi h^2\{\eta^*(2-\eta^*)/2 + (\eta-\eta^*)(1/2 - (\eta+\eta^*)/2)\}\,\frac{\alpha}{\alpha - c}\,w \quad \text{(C-2)}$$

On working out the energy equation we then obtain for the geometrically linear collapse load:

$$q = (h/a)^2 f_b'\{\eta^*(2-\eta^*)/2 + (\eta-\eta^*)(1/2-(\eta+\eta^*)/2)\}\,\frac{4\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \quad \text{(C-3)}$$

Optimization on the basis of $\partial q/\partial\eta$ leads to the condition that $\eta = 1/2$, i.e., the depth of the compressive zone at each yield line is equal to half the depth of the slab. In fact this condition complies with the equilibrium of the sector within the plane of the slab.

The resulting expression for the maximum load that can be carried is:

$$q = \tfrac{1}{2}(h/a)^2 f_b'\{\eta^*(2-\eta^*) + (1/2-\eta^*)^2\}\,\frac{4\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \quad \text{(C-4)}$$

In order to obtain some idea of the loadbearing capacity of an isotropically reinforced flat slab, its strength will, by way of explanatory illustration, be considered with reference to a particular case.

Choose:

$c = 0{,}1 \rightarrow \alpha = 0{,}575$
$f_b' = 18\ \text{N/mm}^2$
$f_a = 400\ \text{N/mm}^2$

ω_a	$q_{el} \times (h/a)^2$	$q_{dome} \times (h/a)^2$	$q_{el+dome} \times (h/a)^2$
0,0	0,0	38,16	38,16
0,005	31,88	23,06	54,94
0,01	60,28	11,79	72,07
0,02	105,49	8,58	113,97
0,025	114,48	0,0	114,48

For the sake of clarity a distinction has been drawn between the elementary contribution (subscript "el") and the contribution due to the restraint forces associated with the dome effect (subscript "dome"). For a reinforcement percentage of 0.5% the contribution of the dome effect is found to be relatively large (72% increase over the elementary strength).

Next, the collapse load including the dome effect will be determined with the method of virtual work. In addition, the straight line will be determined which constitutes the tangent to the geometrically non-linear curve in the load-displacement diagram for the point $w/h = 0$.

The equilibrium system shown in Fig. 11 is adopted as the basis for the analysis ($\eta = 1/2$). The depth of the fully plastic zones at the yield lines is assumed to remain constant during collapse.

The deflection of the positive yield line in the span is chosen as the degree of freedom of the kinematically possible mechanism. Note that the terms which describe the second-order effects are likewise included in the description.

First, the virtual work equation in general terms, as applicable to a framed structure, for example, will be presented:

$$\Sigma M_p \delta\phi + \Sigma N(\theta \delta\theta) = \Sigma F \delta u \tag{C-5}$$

The first two terms can be conceived as the variation of the energy dissipated at the yield lines, while the last term can be conceived as the variation of the work done by the external load. For the local mechanism of a flat slab an expression can be written which in general has the following form:

$$\delta E_d + \delta A_n = \delta A_q \tag{C-6}$$

The above virtual work equation will be applied to the column head mechanism in a flat slab. For the dissipated energy:

$$E_d = 4\pi mp \frac{\alpha}{\alpha - c} w$$

and

$$\delta E_d = 4\pi mp \frac{\alpha}{\alpha - c} \delta w \tag{C-7}$$

For the determination of the variation of the work done by the normal forces at the yield lines it is necessary to distinguish between the radial and the tangential yield lines. The following general relationship holds:

$$\delta A_n = \int\int n_{tt} w_{,t} \delta w_{,t} l \, dA + \int\int n_{rr} w_{,r} \delta w_{,r} l \, dA + \int\int n_{tt} \delta u_{tt} \, dA \qquad \text{(C-8)}$$

where:

$$w_{,t} = \theta = \frac{w}{(\alpha - c)a/2} \quad \text{and} \quad w_{,r} = 0$$

On working this out we obtain:

$$\delta A_n = 2\pi \,(hf_b'/2 - \omega_a f_a h)\, \frac{\alpha}{\alpha - c}\, w\delta w \qquad \text{(C-9)}$$

The variation of the work done by the external load is:

$$\delta A_q = qa^2 \left\{ \frac{\alpha - c - (\alpha^3 - c^3)\pi/12}{\alpha - c} \right\} \delta w \qquad \text{(C-10)}$$

Substitution of (C-7), (C-8) and (C-9) into (C-6) gives the following expression for the collapse load:

$$q = (1/a^2)[mp - (1/2 - \eta^*)(1 - c/\alpha)3/4h^2 f_b'(w/h)]\, \frac{4\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \qquad \text{(C-11)}$$

On substituting $m_p = \frac{1}{2}h^2 f_b'(1/4 + \eta^*)$, i.e., the full plastic moment including dome effect, we obtain:

$$q = \tfrac{1}{2}(h/a)^2 f_b'[(1/4 + \eta^*) - (1/2 - \eta^*)(1 - c/\alpha)(w/h)3/2]\, \frac{4\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \qquad \text{(C-12)}$$

Actually the collapse load is correct only for $w/h = 0$ (since the depth of the compressive zones will change for $w/h > 0$), while the strength is again found to be the geometrically linear collapse load.

It is interesting to note that, with reference to (C-12), the slope of the load-displacement diagram at the origin can be postulated. The collapse load according to (C-12) gives a relationship for a line which is tangential to the geometrically non-linear curve at the point corresponding to $w/h = 0$ in the load-displacement diagram. The difference between the actual curve and the straight line thus obtained is due to the fact that the compressive zone configuration at the yield line has been kept constant. This straight line is indicated in Fig. 12. Its intersection with the horizontal axis is determined from the condition that $q = 0$ in accordance with equation (C-12). This is valid for a ratio w/h as follows:

$$\frac{w}{h} = \frac{1 + 4\eta^*}{4(1 - 2\eta^*)(1 - c/\alpha)3/2} \qquad \text{(C-13)}$$

where:

w = deflection at the positive yield line in the span
h = depth of slab
$\eta^* = \dfrac{\omega_a f_a}{f_b'}$ = ratio of depth of fully plastic zone at yield line to depth of slab

It is to be noted that the collapse load as expressed by equation (C-12) is valid for a flat slab with equal top and bottom reinforcement. Obviously, an expression for the strength (loadbearing capacity) at columns supporting a slab with unequal top and bottom reinforcement can be derived quite simply.

Let ω and ω' be the bottom and top reinforcement percentages respectively. The collapse load is then expressed by (including dome effect, compare (C-4)):

$$q = \tfrac{1}{2}(h/a)^2 f_b' \{\eta^*(2-\eta^*) + (1/2-\eta^*)^2 + \eta^{*\prime}(2-\eta^{*\prime}) + (1/2-\eta^{*\prime})^2\} \cdot \frac{2\pi\alpha}{\alpha - c - (\alpha^3 - c^3)\pi/12} \qquad \text{(C-14)}$$

where:

$$\eta^* = \frac{\omega_a f_a}{f_b'} \quad \text{and} \quad \eta^{*\prime} = \frac{\omega_a' f_a}{f_b'}$$